On the Tangential Displacement of a Surface Point Due to a Cuboid of Uniform Plastic Strain in a Half-Space

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*The elastic solution of a tangentially loaded contact is known as Cerruti's solution. Since the contact surfaces could be easily discretized in small rectangles of uniform shear stress the elastic problem is usually numerically solved by summation of well known integral solution. For soft metallic materials, metals at high temperature, rough surfaces, or dry contacts with high friction coefficient, the yield stress within the material could be easily exceeded even at low normal load. This paper presents the effect of a cuboid of uniform plastic strain in a half-space on the tangential displacement of a surface point. The analytical solutions are first presented. All analytical expressions are then validated by comparison with the finite element method. It is found that the influence coefficients for tangential displacements are of the same order of magnitude as the ones describing the normal displacement (Jacq et al., 2002, "Development of a Three-Dimensional Semi-Analytical Elastic-Plastic Contact Code," ASME J. Tribol., **124**(4), pp. 653–667). This result is of great importance for frictional contact problem when coupling the normal and tangential behaviors in the elastic-plastic regime, such as stick-slip problems, and also for metals and alloys with low or moderate yield stress.* [DOI: 10.1115/1.3197178]

Keywords: contact mechanics, analytical solution, plasticity, friction, Green's function, influence coefficients

1 Introduction

The semi-analytical method (SAM)—which consists of the numerical summation of analytical solutions of elementary problems—requires the determination of integrals also called influence coefficients or Green's functions. For elastic contact problems involving half-spaces, these elementary analytical solutions are known as Boussinesq's, Cerruti's, or Love's solutions. They relate the surface pressure and shear distributions to the subsurface stress state and surface displacements. It was shown for frictional contacts that the coupling between normal and tangential loadings plays a key role in the determination of the surface shear distribution, also known as the stick-slip problem [1–4].

On the other hand, the effect of plasticity was recently introduced in SAM contact solvers [5–17] based on the pioneering work of Jacq et al. [5]. The SAM offers an advantage over the finite element method (FEM) to give a transient three-dimensional solution for nonsmooth surfaces in a reasonable time on a monoprocessor personal computer [18]. However, until now, the tangential displacement of a surface point due to a cuboid of uniform plastic strain in a half-space was not considered since the integral solutions have not yet been determined or at least published. This paper presents the corresponding integrals within the context of contact problem and shows that the influence coefficients for tangential displacements are of the same order of magnitude as the normal displacement. The coupling of normal and tangential effects in contact problems with hardening is now possible. This will permit to apply semi-analytical methods to most frictional contact problems where hardening may occur.

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2 Theoretical Background

A three-dimensional elastic-plastic (EP) rolling contact code based on semi-analytical method was developed and validated [5]. One of the advantages of the method is that it requires the discretization of the contact surface and plastic zone only. Thus, the numerical system is dramatically smaller than the equivalent when using FEM. The use of several numerical methods—including the conjugate gradient method (CGM) to solve the contact problem [19], the discrete convolution fast Fourier transform (DC-FFT) for the numerical summations [20], and the return mapping algorithm for the plasticity loop [9,21]—increases significantly its efficiency in reducing drastically the computing time. The normal contact is solved within the Hertz framework and the surface deformation is expressed as a function of both contact forces and plastic strains. However, the tangential displacements due to plastic strains are neglected and the normal and tangential problems are uncoupled, which gives an accurate solution for frictionless EP contact problem but not for frictional ones, as shown in Ref. [13].

The tangential displacements generated by plastic strains can be expressed by the integration of the reciprocal theorem, and then calculated either numerically or analytically. These will define surface geometry more accurately for frictional problem where tangential displacements are needed, for example, for stick-slip contact problems [22–24], or when high plastic strains are involved. Note that in Refs. [22–24] the normal and tangential contact problems are coupled through what is called the Panagiotopoulos process [25].

3 Normal and Tangential Problems in Elastoplasticity

Both normal and tangential displacements of a surface point due to a cuboid of uniform plastic strain should be carefully considered when solving EP contact problem in the stick/slip regime. The normal displacement was given in an integral form by Chiu [26], which was analytically integrated in Ref. [5]. It is also given

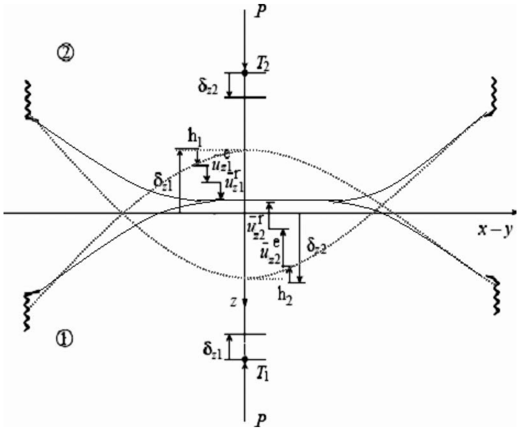

Fig. 1 Residual displacements used in normal problem

in terms of Galerkin vectors in Ref. [27]. The knowledge of normal displacements, see Fig. 1, is sufficient for most frictionless contact problems. For frictional contacts the tangential problem, see Fig. 2, must be solved to correctly define shears in stick-zones. Tangential displacements induced by plasticity are useful in this problem, but have not yet been studied nor considered.

4 Maxwell–Betti Reciprocal Theorem

Consider two independent loads applied to an elastic body of volume Ω and of boundary Γ. The first state $(u,\varepsilon,\sigma,f_i)$ exists with initial strains $\varepsilon°$. The second state is undefined for the moment and will be noted $(u^*,\varepsilon^*,\sigma^*,f_i^*)$. The reciprocal theorem, also know as Maxwell–Betty theorem, expresses an equilibrium of works between both states. It was shown that the theorem with initial strains [5] can be written as

$$-\int_\Gamma u_i^* \cdot \sigma_{ij} \cdot n_j d\Gamma + \int_\Omega f_i \cdot u_i^* d\Omega = -\int_\Gamma u_i \cdot \sigma_{ij}^* \cdot n_j d\Gamma + \int_\Omega f_i^* \cdot u_i d\Omega - \int_\Omega \varepsilon_{ij}^0 \cdot \sigma_{ij}^* d\Omega \tag{1}$$

The reciprocal theorem is now applied to both bodies in contact, where each of them is considered as a half-space Ω whose boundary Γ is loaded on a part $\Gamma c(A)$ and the initial strains occupy a volume $\Omega_p(C)$. Body forces are neglected in both states ($f^*=0$ and $f=0$) and it will be considered that $\mathrm{Tr}(\varepsilon^p)=0$, which is a classical assumption in plasticity. Since $\sigma_{ij}n_j=-p_i$, then Eq. (1) becomes

$$\int_{\Gamma c} u_i^* \cdot p_i d\Gamma = \int_\Gamma u_i \cdot p_i^* d\Gamma - 2\mu \cdot \int_{\Omega p} \varepsilon_{ij}^p \cdot \varepsilon_{ij}^* d\Omega \tag{2}$$

Plastic strains $\varepsilon°$ and pressure distribution p of the real state are known. So, the unknown displacement U can be expressed by

$$U_i(A) = \int_{\Gamma c} u_i^*(M,p^*(A)) \cdot p_i(M) d\Gamma + 2\mu \cdot \int_{\Omega p} \varepsilon_{ij}^p(M) \cdot \varepsilon_{ij}^*(M,p_i^*(A)) d\Omega \tag{3}$$

with M a point of the surface or volume integration. The first part of Eq. (3) is known as Love and Cerruti's term.

The surface displacement of each body can then be expressed as a function of contact loads and of plastic strain existing in the considered body. The authors will now consider only one body in contact with an elastic-plastic behavior; the other one being purely elastic. The formulation could be extended to the case of two elastic-plastic bodies without major difficulties. It is then necessary to express the virtual strains as a function of the virtual contact loads. Note that in what follows both (x_1,x_2,x_3) and (x,y,z) will be used indifferently as a coordinate system. The method and the analytical functions F_{3ij} that describe the normal displacement of a surface point due to a cuboid of uniform plastic strain are recalled in the Appendix to gather all important analytical solutions in a single paper. What now follows focuses on the tangential displacement of a surface point due to a cuboid of uniform plastic strain.

Let us consider a virtual state corresponding to the application of a unit force Q_x along the x-axis applied on the elementary surface area centered at point A, as shown in Fig. 3. The displacements generated are expressed by Cerruti [28]

$$U^*x = \frac{Qx}{4\cdot\pi\cdot G}\cdot\left[\frac{1}{R}+\frac{x^2}{R^3}+(1-2\cdot\upsilon)\cdot\left\{\frac{1}{R+z}-\frac{x^2}{R\cdot(R+z)^2}\right\}\right]$$

$$U^*y = \frac{Qx}{4\cdot\pi\cdot G}\cdot\left[\frac{x\cdot y}{R^3}-(1-2\cdot\upsilon)\cdot\left\{\frac{x\cdot y}{R\cdot(R+z)^2}\right\}\right]$$

$$U^*z = \frac{Qx}{4\cdot\pi\cdot G}\cdot\left[\frac{x\,.\,z}{R^3}+(1-2\cdot\upsilon)\cdot\left\{\frac{x}{R\cdot(R+z)}\right\}\right] \tag{4}$$

The reciprocal theorem is used to express the surface displacements as a function of contact forces and plastic strains within the body under the assumption that $\mathrm{Tr}(\varepsilon^p)=0$ as follows:

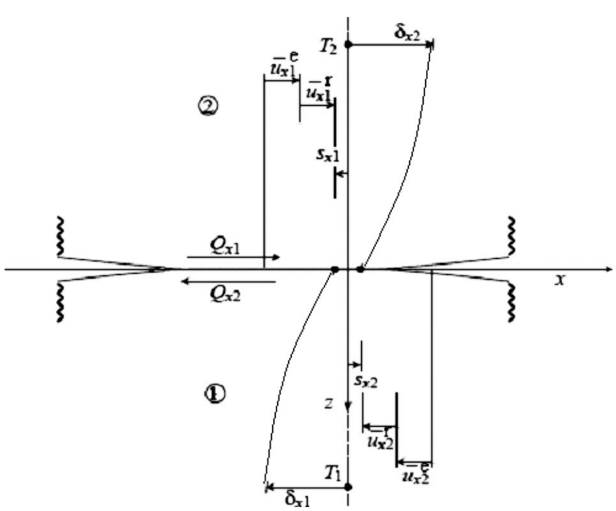

Fig. 2 Residual displacements used in tangential problem

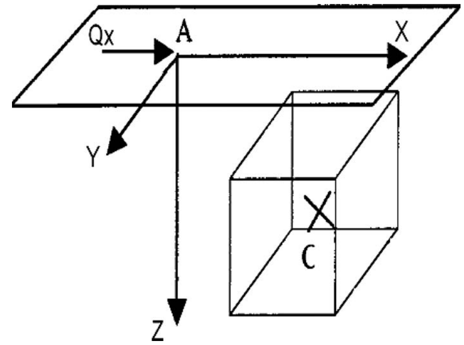

Fig. 3 Cuboid of uniform plastic strain

$$u_1^r(A) = 2\cdot\mu\cdot\sum_{n=1}^{Nv}\varepsilon_{ij}^p(C_n)\cdot\int_{\Omega p}\varepsilon_{1ij}^*(A,M)d\Omega = \sum_{n=1}^{Nv}\varepsilon_{ij}^p\cdot D_{kij}(A,C_n) \quad (5)$$

with U^* defined by Eq. (4) and $\varepsilon^*_{1ij}=\frac{1}{2}\cdot(U^*_{i,j}+U^*_{j,i})$

The residual displacements are functions of plastic strain $\varepsilon^p{}_{ij}$, Poisson's ratio ν, and cuboid location $(c1,c2,c3)$ and size $(2\Delta x,2\Delta y,2\Delta z)$.

$$\begin{aligned}D_{ijk} &= F_{ijk}(c1+\Delta x,c2+\Delta y,c3+\Delta z) - F_{ijk}(c1+\Delta x,c2+\Delta y,c3 \\ &\quad -\Delta z) - F_{ijk}(c1+\Delta x,c2-\Delta y,c3+\Delta z) - F_{ijk}(c1-\Delta x,c2 \\ &\quad +\Delta y,c3+\Delta z) + F_{ijk}(c1+\Delta x,c2-\Delta y,c3-\Delta z) + F_{ijk}(c1 \\ &\quad -\Delta x,c2-\Delta y,c3+\Delta z) + F_{ijk}(c1-\Delta x,c2+\Delta y,c3-\Delta z) \\ &\quad - F_{ijk}(c1-\Delta x,c2-\Delta y,c3-\Delta z)\end{aligned} \quad (6)$$

The D_{1ij} functions may be derived from Eqs. (4) and (5) in an integral form

$$D111 = \frac{1}{2\cdot\pi}\cdot\int\int\left[\frac{1}{\rho}+\frac{x^2}{\rho^3}+(1-2\cdot\upsilon)\left\{\frac{1}{\rho+z} - \frac{x^2}{\rho\cdot(\rho+z)^2}\right\}\right]dydz$$

$$D122 = \frac{1}{2\cdot\pi}\cdot\int\int\left[\frac{x\cdot y}{\rho^3}-(1-2\cdot\upsilon)\frac{x\cdot y}{\rho\cdot(\rho+z)^2}\right]dxdz$$

$$D133 = \frac{1}{2\cdot\pi}\cdot\int\int\left[\frac{x\cdot z}{\rho^3}+(1-2\cdot\upsilon)\frac{x}{\rho\cdot(\rho+z)}\right]dxdy$$

$$D112 = \frac{1}{4\cdot\pi}\cdot\left[\int\int\left[\frac{1}{\rho}+\frac{x^2}{\rho^3}+(1-2\cdot\upsilon)\left\{\frac{1}{\rho+z} - \frac{x^2}{\rho\cdot(\rho+z)^2}\right\}\right]dxdz + \int\int\left[\frac{x\cdot y}{\rho^3}-(1-2\cdot\upsilon)\frac{x\cdot y}{\rho\cdot(\rho+z)^2}\right]dydz\right]$$

$$D113 = \frac{1}{4\cdot\pi}\cdot\left[\int\int\left[\frac{1}{\rho}+\frac{x^2}{\rho^3}+(1-2\cdot\upsilon)\left\{\frac{1}{\rho+z} - \frac{x^2}{\rho\cdot(\rho+z)^2}\right\}\right]dxdy + \int\int\left[\frac{x\cdot z}{\rho^3}+(1-2\cdot\upsilon)\frac{x}{\rho\cdot(\rho+z)}\right]dydz\right]$$

$$D123 = \frac{1}{4\cdot\pi}\cdot\left[\int\int\left[\frac{x\cdot y}{\rho^3}-(1-2\cdot\upsilon)\frac{x\cdot y}{\rho\cdot(\rho+z)^2}\right]dxdy + \int\int\left[\frac{x\cdot z}{\rho^3}+(1-2\cdot\upsilon)\frac{x}{\rho\cdot(\rho+z)}\right]dxdz\right] \quad (7)$$

The analytical expressions for functions $F1jk$ below were obtained by analytical integration as done by Jacq et al. [5] for functions $F3jk$. The latter functions, which describe the normal displacements at the surface generated by a unique plastic zone, are recalled in the Appendix along with some details on the method.

$$\begin{aligned}F111 &= \frac{1}{2\pi}\cdot\Bigg[z\log(R+y)+y\log(R+z) + 2\cdot x\cdot\arctan\left(\frac{R+y+z}{x}\right)+x\cdot\arctan\left(\frac{y\cdot z}{x\cdot R}\right)+(1 \\ &\quad -2\cdot\upsilon)\cdot\left[2\cdot x\cdot\arctan\left(\frac{R+y+z}{x}\right)+z\log(R+y) + \frac{1}{2}\cdot y\log(R+z)-\frac{z\cdot y}{2\cdot(R+z)}\right]\Bigg]\end{aligned}$$

$$F122 = \frac{1}{2\pi}\cdot\left[-y\cdot\log(R+z)+(1-2\cdot\upsilon)\cdot y\cdot\left[\frac{z}{2\cdot(R+z)}+\frac{1}{2}\cdot\log(R+z)\right]\right]$$

$$F133 = \frac{1}{2\pi}\cdot\left[-2\cdot\upsilon\cdot z\log(R+y)+(1-2\cdot\upsilon)\cdot\left[2\cdot x\cdot\arctan\left(\frac{R+y+z}{x}\right)+y\log(R+z)\right]\right]$$

$$F112 = \frac{1}{\pi}\cdot\left[z\log(R+x)+2\cdot y\cdot\arctan\left(\frac{R+x+z}{y}\right) + \frac{(1-2\cdot\upsilon)}{2}\cdot\left[x\log(R+z)+\frac{z\cdot x}{R+z}\right]\right]$$

$$F113 = \frac{1}{\pi}\cdot\left[y\log(R+x)+2\cdot z\cdot\arctan\left(\frac{x+y+R}{z}\right)\right]$$

$$F123 = -\frac{R}{\pi} \quad (8)$$

with $R=\sqrt{x^2+y^2+z^2}$.

Actually, $D1jk$ is a coefficient of influence defining the tangential displacement along the X-direction of a cuboid located at $(c1,c2,c3)$ and of size $(2\Delta x,2\Delta y,2\Delta z)$ in which strain ε_{ij} is considered equal to one.

$F2jk$ functions can be easily found by circular permutation of indices. The tangential residual displacements of the surface due to plastic strains existing in the half-space can now be computed using the relations described above.

5 Finite Element Model

A comparison with a finite element analysis (FEA) performed with the commercial FE package ABAQUS V6.7 will be performed to validate the analytical solutions and check the analytical methods' accuracy for both $F1jk$ and $F3jk$ solutions. The equivalent problem solved by the FE method is presented below.

The use of symmetrical and antisymmetrical boundaries permits to limit the FE model to one-quarter of a parallelepiped (see Fig. 4). One-quarter of a cuboid of uniform plastic strain is set below the surface in such a manner that its axis of symmetry coincides with the vertical axis Z. Symmetrical or antisymmetrical boundary conditions (BCs) are used for $X=0$ and $Y=0$ planes, but depending on plastic strains imposed, those boundaries will change, as shown in Table 1. The upper face is naturally free. On the three other external faces that are far away from the plastic cuboid, free and fully constrained boundaries have been used alternatively. No difference has been observed on the results, so the meshed body could be considered as a quarter of a half-space.

Using those symmetries, the problem is 60 units wide by 60 units high. The volume near the plastic cuboid and above at the surface has been refined with 0.5-sized elements. Therefore one-quarter of the half-space is meshed with only 30,000 C3D8 ele-

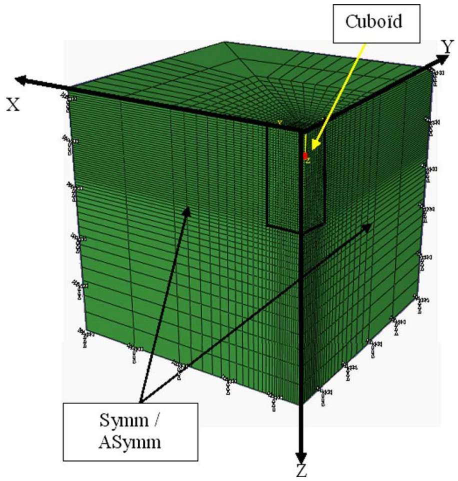

Fig. 4 Finite element model

ments. While the "plastic zone" is supposed of unit size in the full problem, two refined cuboids of total dimension $0.5\times0.5\times1$ are used in this quarter problem. The center of the equivalent cuboid is here located at a point five units below the surface.

The same elastic properties are used for both the elastic and plastic zones. The Young modulus choice does not matter while it has no influence on strain and displacement in a dimensionless form. Thus it is arbitrary taken equal to 1 MPa. On the other hand, Poisson's ratio is important and taken equal to 0.3 since it is considered as a nonspecific case.

The model is based on an equivalent approach using thermo-elastic properties for this plastic zone to create a thermal expansion similar to the plastic strain required. For this reason, anisotropic thermal expansion properties are defined for the two "plastic elements." During step 1 a predefined thermal field $T=1\,^\circ\mathrm{C}$ creates the additional elastic strain desired

$$\varepsilon_{ij}^{T} = \alpha_{ij}^{T}\cdot T \qquad (9)$$

FE analyses have been performed under the assumption of linear elasticity ("perturbation" option for ABAQUS) as done in the analytical modeling.

The assumption of incompressibility ($\mathrm{Tr}(\varepsilon^{p})=0$) of the plastic volume used in the analysis, see Eq. (5), to derivate the analytical expressions in Eqs. (8) and (A5), should be also verified when imposing the equivalent anisotropic thermal expansions. For this reason the diagonal components of the strain tensor should not be imposed individually. However, since the problem is linear, the effect of each tensor component could be superimposed resulting in fine in the validation of each term individually. The five equivalent thermal loadings finally used to validate the analytical solutions are given in Table 1.

6 Comparison Between FEA and Analytical Results

Let us consider a cuboid of uniform strain of dimension $2b\times2b\times2b$ ($b\times b\times2b$ for the quarter FE model) with its center located at a distance $C_3=Xo/(2\cdot b)$ below the surface. The surface displacements reduce to Eq. (10) that can be easily plotted knowing the analytical solutions expressed in Eq. (8).

$$u_k^{r}(A) = \varepsilon_{ij}^{p}\cdot D_{kij}(A,C_n) \qquad (10)$$

As previously stated all FE simulations will be performed for a cuboid of uniform strain located at depth $C_3=Xo/(2\cdot b)=5$ in a dimensionless form. A comparison of the numerical results obtained by FEA and those derived from the analytical expressions is presented in Figs. 5–10. The results are given for a surface profile first along the X- or Y-axis (Figs. 5–8), and second for the first bisectrice in the plane (O,X,Y), see Figs. 9 and 10, since due

Table 1 Boundary conditions of the FEM

Case No.	Strains	Symmetries
(1)	$\varepsilon_{11}^{p}=-\varepsilon_{22}^{p}=0.001$	XSym/YSym
(2)	$\varepsilon_{11}^{p}=-\varepsilon_{33}^{p}=0.001$	XSym/YSym
(3)	$\varepsilon_{12}^{p}=0.001$	XASym/YASym
(4)	$\varepsilon_{13}^{p}=0.001$	XASym/YSym
(5)	$\varepsilon_{23}^{p}=0.001$	XSym/YASym

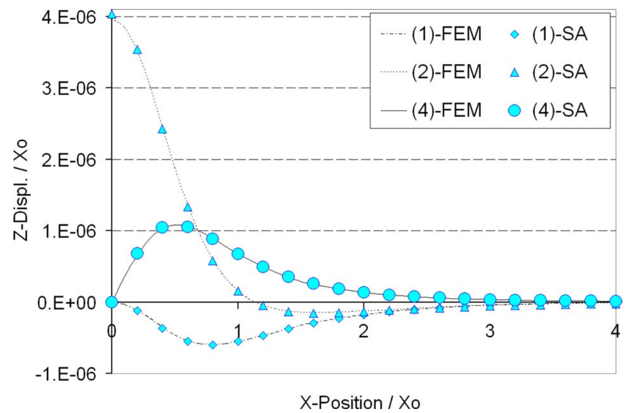

Fig. 5 Normal displacements U_Z along the X-axis ($Xo/2b=5$)

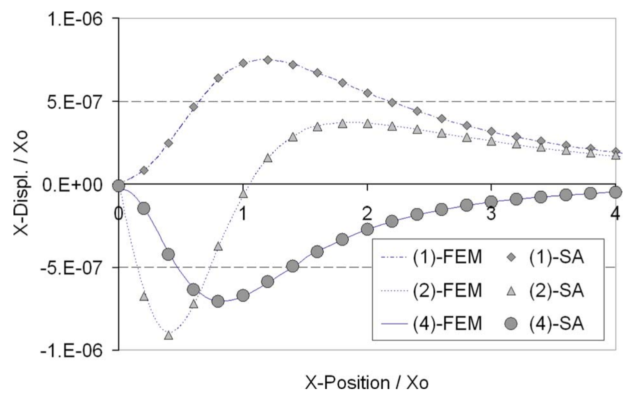

Fig. 6 Tangential displacements U_X along the X-axis ($Xo/2b=5$)

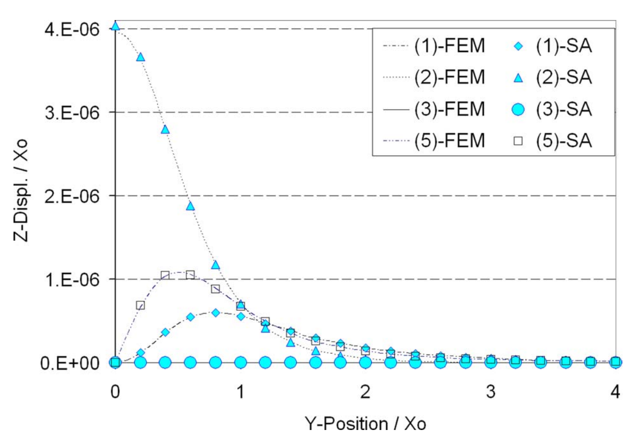

Fig. 7 Normal displacements U_Z along the Y-axis ($Xo/2b=5$)

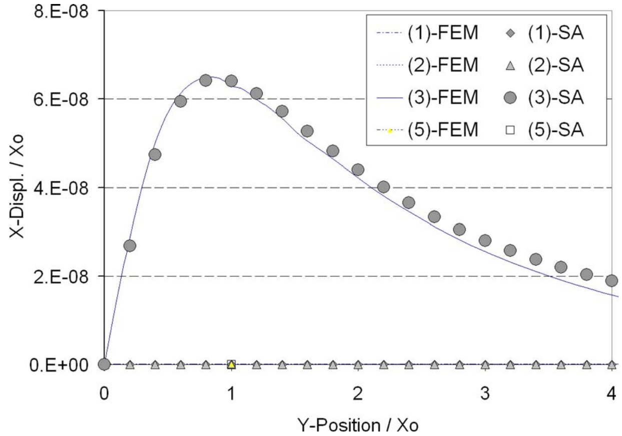

Fig. 8 Tangential displacements U_X along the *Y*-axis ($Xo/2b=5$)

to the (anti-)symmetrical properties imposed some *Fijk* functions are equal to zero.

Finally it is found a remarkably good agreement between the numerical (FEA) and the analytical solutions proposed by the authors. The only noticeable difference is found in Fig. 8 for the in-plane displacement due to the strain component $\varepsilon^p{}_{12}$. This difference is due to the failure of the boundary conditions in reproducing the half-space assumption for a thermal distortion of the cuboid in the plane parallel to the surface, which appears far away from the origin when moving closer to the domain boundaries.

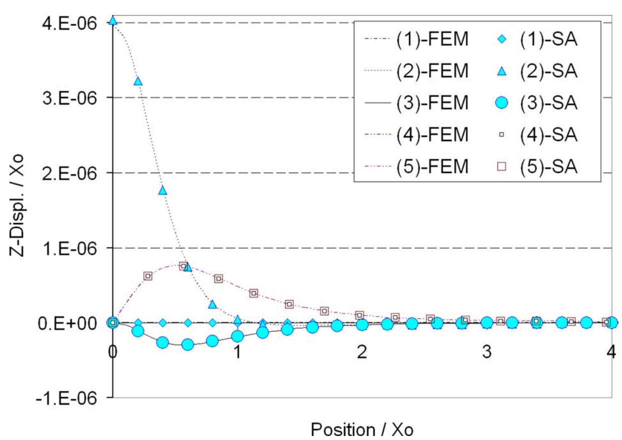

Fig. 9 Normal displacements U_Z over an angle of 45 deg ($Xo/2b=5$)

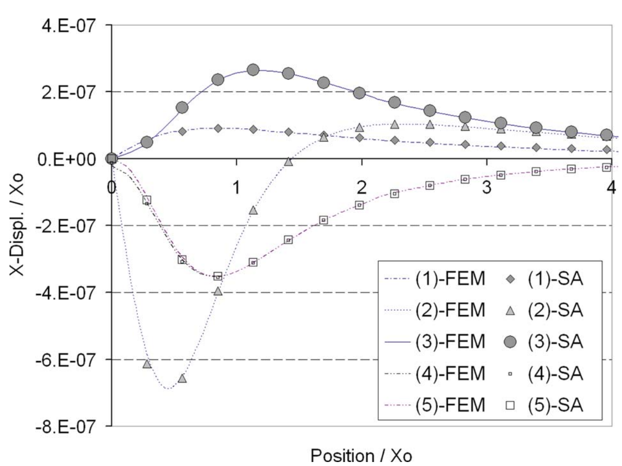

Fig. 10 Tangential displacements U_X over an angle of 45 deg ($Xo/2b=5$)

7 Conclusion

Contact solvers based on the semi-analytical method need elementary analytical solutions to compute. The Love [29] and Cerruti [28] solutions used in elastic contact solver are elementary solutions relating the surface pressure and shear distributions to the subsurface stress state and surface displacements. New elementary solutions are required when plasticity is involved. While Chiu [26] described the effect of a unique plastic zone on the residual stress and strain states, Jacq et al. [5] related analytically the subsurface plastic strains to the normal surface displacements. All these elementary solutions were used in a three-dimensional semi-analytical elastic-plastic contact code [5]. A special attention to the tangential effects involved by plasticity should be paid since it has not been properly considered until now.

This paper gives the analytical expressions that relate the components of the strain tensor of an incompressible cuboid of uniform strain on the tangential displacements of any point of the free surface of a half-space. Both normal and tangential solutions have been compared to a finite element analysis and a very good agreement was found, which validates the analytical solutions. The next step will be the implementation of the analytical solutions into an elastic-plastic contact solver to investigate frictional contact problems.

Considering FEM nonlinearities encountered in contact problems, which generate cumbersome computations, models based on semi-analytical methods—with such new ingredients to account for plasticity—could be an interesting alternative.

Acknowledgment

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Nomenclature

C_n = cuboid of size $(2\Delta x, 2\Delta y, 2\Delta z)$ and center point (x, y, z)

$D_{kij}(A, C_n)$ = influence function of the ij strain component of cuboid C_n on the component k of the residual displacements at the surface point A

$u^r{}_k(A)$ = residual displacement of a surface point A in the k-direction

$u^e{}_k(A)$ = elastic displacement of a surface point A in the k-direction

$u^*{}_k(M)$ = displacements created by a unit force applied along the k-axis at the origin of the surface, and calculated at the subsurface point M

δ_{ki} = rigid body displacement along the k-direction for body i

$\varepsilon^p{}_{ij}(C_n)$ = plastic strain component

$\varepsilon^*{}_{kij}(A, C_n)$ = ij strain component created in the cuboid C_n by a unit force along the k-axis applied at the surface point A

Ω_p = plastic volume

Appendix: Normal Displacements

Let us consider a virtual state corresponding to the application of a unit force along the x-axis applied on the elementary surface area centered at point A (see Fig. 11) [5].

The displacements generated are expressed by Boussinesq (see Ref. [1]).

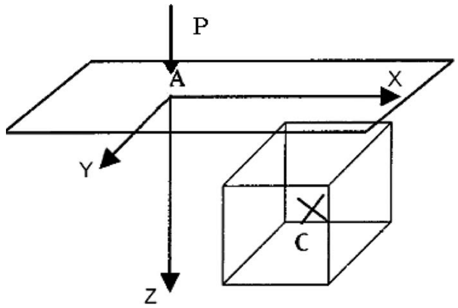

Fig. 11 Cuboid of uniform plastic strain

$$U^{*}x=\frac{P}{4\cdot\pi\cdot G}\cdot\left[\frac{x\cdot z}{R^3}-(1-2\cdot\upsilon)\cdot\frac{x}{R\cdot(R+z)}\right]$$

$$U^{*}y=\frac{P}{4\cdot\pi\cdot G}\cdot\left[\frac{y\cdot z}{R^3}-(1-2\cdot\upsilon)\cdot\frac{y}{R\cdot(R+z)}\right]$$

$$U^{*}z=\frac{P}{4\cdot\pi\cdot G}\cdot\left[\frac{z^2}{R^3}+\frac{2\cdot(1-\upsilon)}{R}\right] \tag{A1}$$

The reciprocal theorem is used to express the surface displacements as a function of contact forces and plastic strains within the body under the assumption that the plastic volume is incompressible, i.e., $\mathrm{Tr}(\varepsilon^p)=0$

$$u_3^r(A)=2\cdot\mu\cdot\sum_{n=1}^{Nv}\varepsilon_{ij}^p(C_n)\cdot\int_{\Omega p}\varepsilon_{3ij}^{*}(A,M)d\Omega=\sum_{n=1}^{Nv}\varepsilon_{ij}^p\cdot D_{3ij}(A,C_n) \tag{A2}$$

Consequently the residual displacements are functions of plastic strain ${\varepsilon^p}_{ij}$, Poisson's ratio ν, and location $(c1,c2,c3)$ and size of the cuboid of uniform plastic strain $(2\Delta x,2\Delta y,2\Delta z)$

$$\begin{aligned}D_{3ij}&=F_{3ij}(c1+\Delta x,c2+\Delta y,c3+\Delta z)-F_{3ij}(c1+\Delta x,c2+\Delta y,c3\\&\quad-\Delta z)-F_{3ij}(c1+\Delta x,c2-\Delta y,c3+\Delta z)-F_{3ij}(c1-\Delta x,c2\\&\quad+\Delta y,c3+\Delta z)+F_{3ij}(c1+\Delta x,c2-\Delta y,c3-\Delta z)+F_{3ij}(c1\\&\quad-\Delta x,c2-\Delta y,c3+\Delta z)+F_{3ij}(c1-\Delta x,c2+\Delta y,c3-\Delta z)\\&\quad-F_{3ij}(c1-\Delta x,c2-\Delta y,c3-\Delta z)\end{aligned} \tag{A3}$$

The D_{3ij} functions may be derived from Eqs. (A1) and (A2) in an integral form

$$D311=\frac{1}{2.\pi}\cdot\int\int\left[\frac{x\cdot z}{\rho^3}-(1-2\cdot\upsilon)\frac{x}{\rho\cdot(\rho+z)}\right]dydz$$

$$D322=\frac{1}{2.\pi}\cdot\int\int\left[\frac{y\cdot z}{\rho^3}-(1-2\cdot\upsilon)\frac{y}{\rho\cdot(\rho+z)}\right]dxdz$$

$$D333=\frac{1}{2\cdot\pi}\cdot\int\int\left[\frac{z^2}{\rho^3}+\frac{2\cdot(1-\upsilon)}{\rho}\right]dxdy$$

$$\begin{aligned}D312=\frac{1}{4\cdot\pi}\cdot\Bigg[&\int\int\left[\frac{x\cdot z}{\rho^3}-(1-2\cdot\upsilon)\frac{x}{\rho\cdot(\rho+z)}\right]dxdz\\&+\int\int\left[\frac{z^2}{\rho^3}+\frac{2\cdot(1-\upsilon)}{\rho}\right]dydz\Bigg]\end{aligned}$$

$$\begin{aligned}D313=\frac{1}{4\cdot\pi}\cdot\Bigg[&\int\int\left[\frac{x\cdot z}{\rho^3}-(1-2\cdot\upsilon)\frac{x}{\rho\cdot(\rho+z)}\right]dxdy\\&+\int\int\left[\frac{y\cdot z}{\rho^3}-(1-2\cdot\upsilon)\frac{y}{\rho\cdot(\rho+z)}\right]dydz\Bigg]\end{aligned}$$

$$\begin{aligned}D323=\frac{1}{4\cdot\pi}\cdot\Bigg[&\int\int\left[\frac{z^2}{\rho^3}+\frac{2\cdot(1-\upsilon)}{\rho}\right]dxdy+\int\int\left[\frac{y\cdot z}{\rho^3}-(1\right.\\&\left.-2\cdot\upsilon)\frac{y}{\rho\cdot(\rho+z)}\right]dxdz\Bigg]\end{aligned} \tag{A4}$$

The F_{3ij} functions were analytically integrated by Jacq et al. [5] as

$$F311=\frac{1}{\pi}\cdot\left[-\upsilon\cdot x\log(y+R)-(1-2\cdot\upsilon)\cdot z\arctan\left(\frac{y+z+R}{x}\right)\right]$$

$$F322=\frac{1}{\pi}\cdot\left[-\upsilon\cdot y\log(x+R)-(1-2\cdot\upsilon)\cdot z\arctan\left(\frac{x+z+R}{y}\right)\right]$$

$$\begin{aligned}F333=\frac{1}{\pi}\cdot\Bigg[&(1-\upsilon)\cdot\left[2\cdot z\cdot\arctan\left(\frac{R+y+x}{z}\right)+x\log(R+y)\right.\\&\left.+y\log(R+x)\right]+\frac{z}{2}\cdot\arctan\left(\frac{x\cdot y}{z\cdot R}\right)\Bigg]\end{aligned}$$

$$F312=\frac{1}{\pi}\cdot[-2\cdot\upsilon\cdot R-(1-2\cdot\upsilon)\cdot z\log(z+R)]$$

$$F313=\frac{1}{\pi}\cdot\left[2\cdot x\arctan\left(\frac{y+z+R}{x}\right)+y\log(z+R)\right]$$

$$F323=\frac{1}{\pi}\cdot\left[2\cdot y\arctan\left(\frac{x+z+R}{y}\right)+x\log(z+R)\right] \tag{A5}$$

with $R=\sqrt{x^2+y^2+z^2}$.

Normal residual displacements of the surface due to plastic strains existing in the half-space can now be computed using the relations described above.

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